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Abstract : The local behavior of a function near a Hölder singularity located at a point x_0 can be finely described by means of its 2-microlocal domain. We solve the issue of prescribing this domain. A more compact description can be given using two parameters, the Hölder exponent and the chirp exponent. We give a necessary condition satisfied by this couple of exponents considered as functions of x_0.

1 Introduction

Let α be a positive real number and $x_0 \in \mathbf{R}$; a function $f : \mathbf{R} \to \mathbf{R}$ is $C^\alpha(x_0)$ if there exists a polynomial P of degree less than α such that

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^{\alpha.} \tag{1}$$

The *Hölder exponent* $h_f(x_0)$ is the supremum of all the values of α such that (1) holds. $h(.)$ thus defines a function, the pointwise values of which describe the local variations of the irregularity of the function f. The Hölder exponent, as a measure of pointwise regularity of functions, is a powerful tool in many applications such as image analysis or signal analysis [9] [4]. The issue of prescribing $h_f(x)$ has been solved in [6] and [3]: The class of all admissible functions $h_f(x)$ (when $f \in C^0(\mathbf{R})$) coincides with the class of lower limits of continuous functions.

However, characterizing the regularity with the sole Hölder exponent has several drawbacks. It does not give any information about the oscillatory behavior of the function near the Hölder singularities; furthermore $h_f(x)$ is not stable under the action of (pseudo) differential operators. This precludes, for instance, the use of the Hilbert transform, commonly used for estimation on sampled signals. The 2-microlocal spaces $C^{s,s'}_{x_0}$ [5, 8] have been introduced by Bony [2] to cope with such issues in the frame of p.d.e. analysis. The rough idea is to incorporate knowledge about the regularity "in a neighborhood of the point". These spaces can be defined by decay conditions on the wavelet coefficients as follows, see [5].

If the $\psi_{j,k} = 2^{j/2}\psi(2^j x - k)$ form an orthonormal basis of $L^2(\mathbf{R})$, with ψ in the Schwartz class, we define the wavelet coefficients of f by

$$c_{j,k} = 2^j \int f(x)\psi(2^j x - k)dx$$

(note that we do not use a L^2 normalization here).

Definition 1.1 *A distribution F belongs to the 2-microlocal space $C^{s,s'}_{x_0}$ if its wavelet coefficients satisfy, for $|x_0 - k2^{-j}|$ close enough to 0,*

$$\exists C > 0 : \ |c_{j,k}| \leq C 2^{-j(s+s')} \left(2^{-j} + |k2^{-j} - x_0|\right)^{-s'}.$$

It follows from this definition that the *2-microlocal domain of F at x_0*, i.e. $\left\{(s, s') : F \in C^{s,s'}_{x_0}\right\}$, is convex and $F \in C^{s,s'}_{x_0} \Longrightarrow F \in C^{s-\epsilon,s'+\epsilon}_{x_0}, \forall \epsilon > 0$.

This set is thus completely determined by the function

$$S(\sigma, x_0) = \sup\left\{s \,:\, F \in C^{s,\sigma-s}_{x_0}\right\}.$$

In addition, $\sigma \to S(\sigma, x_0)$ is decreasing and concave. In Section 3 we will show that, conversely, all such functions are functions $S(\sigma, x_0)$ associated with two-microlocal frontiers *for a fixed point* x_0. Note that this result has been proved independently using a different method by Y.Meyer in [10]. This mathematical result should also lead to improved estimation methods and new tools for signal processing.

The 2-microlocal domains for all values of x_0 clearly give a very rich information which is superfluous in some applications [1]. It is sometimes sufficient to deal with two numbers: the usual Hölder exponent, and another exponent $\beta(x_0)$ which measures the oscillatory behavior of f near x_0, and is defined as follows.

Definition 1.2 *Let $F^{(-n)}$ a n-th order primitive of F. F is called a (h, β)-type chirp at x_0 if*

$$\forall n \in \mathbf{N} \,:\, F^{(-n)} \in C^{h+n(1+\beta)}_{x_0}.$$

The simplest example of a (h, β) chirp at x_0 is supplied by the function $|x - x_0|^h sin(\frac{1}{|x-x_0|^\beta})$. The interior of the set of points (h, β) such that a function f is an (h, β) type chirp at x_0 is always a domain of the form $h < h_0$, $\beta < \beta_0$; this property, proved in [7], allows to define the *chirp exponent at* x_0 $\beta(x_0)$ as this real number β_0.

In sharp contrast with the problem of the prescription of the sole Hölder exponent, we show in Section 2 that the couple $(h(x_0), \beta(x_0))$ must satisfy a very strong a priori requirement: The function $\beta(x)$ vanishes where it is continuous.

The following inclusions relate the 2-microlocal and Hölder spaces, [5]

$$C^{s,s'}_{x_0} \subset C^{s}_{x_0} \subset C^{s,-s}_{x_0}, \;\; \forall (s + s') > 0. \tag{2}$$

It follows that the chirp exponents of f at x_0 are (h, β) if and only if the boundary of the 2-microlocal domain of f at x_0 is above the half-line starting at $(s, s') = (h, -h)$ of slope $-(\beta + 1)/\beta$, and if this property no more holds for larger values of h or β (see figure 1).

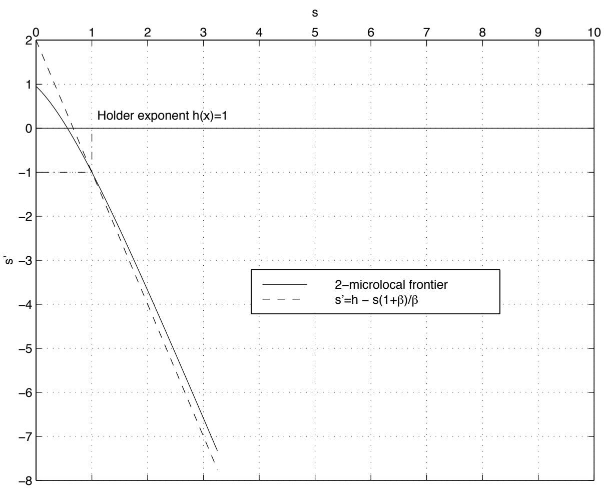

Figure 1: Relation between 2-microlocal domain and chirps exponents

2 A condition on chirps exponents

In this section, we show that it is not possible, in general, to prescribe arbitrarily the chirp exponents of a function independently at each point.

Proposition 2.1 *Let F be a function whose Hölder exponent $h(x)$ satisfies*

$$\text{for all } x \quad 0 < h \leq h(x) \leq H < +\infty;$$

then the chirp exponent $\beta(x)$ vanishes on a dense set of points.

Proof: Let I be a non empty interior interval. Assume $\beta(x) > 0$ for a set $E \subset I$, $\overline{E} = I$. Let $\epsilon > 0$; for any $x \in E$, there exists a sequence $(j_n, k_n) \to (+\infty, x)$ such that

$$|c_{j_n,k_n}| \geq |x - k_n 2^{-j_n}|^{h(x)+\epsilon} \geq |x - k_n 2^{-j_n}|^{H+\epsilon} \tag{3}$$

Moreover

$$\begin{aligned} |x - k_n 2^{-j_n}| &\geq (2^{-j_n})^{\frac{1}{1+\beta(x)}+\epsilon} \\ &\geq (2^{-j_n})^{1+\epsilon}. \end{aligned} \tag{4}$$

Estimate (3) follows from (2); and (4) holds because, if $\beta \neq 0$, $(c_{j,k})$ is rapidly decreasing as a function of 2^{-j} in any domain $2^{-j} \geq |x - k2^{-j}|^{1+\beta-\epsilon}$. Thus,

$$|c_{j_n,k_n}| \geq (2^{-j_n})^{(1+\epsilon)(1+H)} \tag{5}$$

Let $E_{j_n} = \{k_n 2^{-j_n}$ such that (5) holds$\}$ and $F_j = \bigcup_{j_n > j} E_{j_n}$.
For all $x \in E$, the sequence $(k_n 2^{-j_n})$ constructed above converges to x. We construct an $x_0 \in I$ for which the coefficients $c_{j,k}$ verify (5) in the cone $|x_0 - k2^{-j}| \leq 2.2^{-j}$ in the following way : Let $c_{j,k}$ verify (5). We choose (j', k') such that $j' > j$ and $k' 2^{-j'} \in [k2^{-j} - 2^{-j}, k2^{-j} + 2^{-j}]$. This is possible because $\overline{F_j} = I, \forall j$. Carrying on, we construct a sequence of points converging to a point x_0 for which $\beta(x_0) = 0$.

3 2-microlocal prescription at one point

In this section, we deal with the prescription of the function $\sigma \to S(\sigma, x_0)$ for a fixed x_0. The case where $S(\sigma, x_0)$ is a line can be excluded since for the chirps $|x - x_0|^h \sin(|x - x_0|^{-\beta})$, $S(\sigma, x_0) = h(x_0) - \beta(x_0)\sigma$.

Proposition 3.1 *Let $S(\sigma)$ be a concave and decreasing function defined on* $\mathbf{R}$. *Assume that $S(\sigma)$ is not a line. Then, the wavelet coefficients*

$$c_{j,k} = \inf_{\sigma} 2^{-j\sigma}(2^{-j} + |2^{-j}k - x_0|)^{S(\sigma)-\sigma} \tag{6}$$

define a distribution F, the 2-microlocal frontier at x_0 of which is $S(\sigma)$.

Proof: The points $(S(\sigma), \sigma - S(\sigma))$ are clearly contained in the 2-microlocal domain of F. For the optimality, we will assume that, without loss of generality, $x_0 = 0$. For sake of simplicity, let $k \geq 0$:

$$\begin{aligned} u_{j,k} &= ln\left(2^{-j}(k+1)\right) \\ v_{j,k} &= \frac{ln\,(2^{-j})}{u_{j,k}} - 1 = -\frac{ln\,(k+1)}{ln\,(2^{-j}(k+1))} \end{aligned}$$

The 2-microlocal condition, for $k > 0$, at 0

$$|c_{j,k}| \leq C 2^{-j\sigma}(2^{-j} + k2^{-j})^{S(\sigma)-\sigma}$$

can be written

$$ln\left(|c_{j,k}|\right) \leq ln(C) + u_{j,k}.(\sigma v_{j,k} + S(\sigma)), \tag{7}$$

and (6) becomes

$$ln\left(c_{j,k}\right) = u_{j,k}.\sup_{\sigma}(\sigma v_{j,k} + S(\sigma)) = u_{j,k}.\left[-S(.)\right]^{*}\left(v_{j,k}\right) \tag{8}$$

where $f^*(\sigma)$ denotes the Legendre-Fenchel transform of f.

Let $l(\vartheta) = [-S(.)]^*(\vartheta)$. As $(-S(.))$ is a continuous convex function, $l^*(\sigma) = -S(\sigma)$, and thus

$$\forall\sigma, \forall\eta > 0,\ \exists\vartheta_{\sigma,\eta}\ l\left(\vartheta_{\sigma,\eta}\right) \leq \sigma.\vartheta_{\sigma,\eta} + S(\sigma) + \eta. \tag{9}$$

Moreover, $\vartheta_{\sigma,\eta} > 0$, as $S(.)$ is decreasing.
Suppose now that there exist σ and ϵ such that $F \in C_0^{S(\sigma)+\epsilon,\sigma-S(\sigma)-\epsilon}$. Relations (7) and (8) imply that

$$\exists C > 0, \forall(j,k):\quad u_{j,k}l\left(v_{j,k}\right) \leq ln(C) + u_{j,k}\left(\sigma.v_{j,k} + S(\sigma) + \epsilon\right),$$

which can be rewritten

$$l\left(v_{j,k}\right) \geq \frac{ln(C)}{u_{j,k}} + \sigma.v_{j,k} + S(\sigma) + \epsilon. \tag{10}$$

Let $\vartheta_{\sigma,\epsilon/2}$ be defined as above. Since $\vartheta_{\sigma,\epsilon/2} > 0$, there exist $(j(n), k(n))$ such that

$$\lim_{n\to\infty}\left(u_{j(n),k(n)}, v_{j(n),k(n)}\right) = \left(-\infty, \vartheta_{\sigma,\epsilon/2}\right). \tag{11}$$

As $S(.)$ is not a line, $[-S(.)]^*$ is continuous on the right or on the left at each point of its definition domain. Thus, applying (10) for this sequence and taking the limit given by (11),

$$l\left(\vartheta_{\sigma,\epsilon/2}\right) \geq \sigma.\vartheta_{\sigma,\epsilon/2} + S(\sigma) + \epsilon$$

which contradicts (9).

N.B. While the above local construction can obviously be made around a finite number of points, it follows from Section 2 that the 2-microlocal domain cannot be prescribed everywhere.

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